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ORIGINAL RESEARCH ARTICLE

Bayesian and non-Bayesian analysis for the lifetime performance index based on generalized order statistics from Pareto distribution

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ABSTRACT

Modern businesses depend on efficient management and evaluation of product quality performance to assure that they are on the right track, and process capability analysis is used to gauge business performance in practice. Consequently, the lifetime performance index (LPI) C_L, where L is the lower specification limit, is used to gauge a process potential and performance. This paper examines distinct estimators of C_L under Pareto distribution using generalized order statistics (GOS), which is very helpful in a variety of real-world applications. Results for progressive type II censoring (PTIIC) and first-failure censoring are two particular situations. Using symmetric and asymmetric loss functions, the Bayesian estimator was built, then utilized to produce the C_L hypothesis testing technique. A simulation study and real data analysis have been investigated to study the behavior of different estimates for C_L under different schemes, namely PTIIC and the progressive first failure censored scheme.

Keywords: generalized order statistics; Pareto distribution; Bayesian estimator; lifetime performance index

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1. Introduction

The main purpose of the economic field is to meet the requirements of the customer, and one of the main requirements of the customer is the lifetime of the product; a longer lifetime means a better product. Montgomery[1] proposed the process capability index C_L for evaluating the lifetime performance.

$$C_L = \frac{\mu - L}{\sigma} \quad (1)$$

where μ, L and σ, are the process mean, the lower specification limit, and the process standard deviation, respectively. A larger C_L value indicates a better process quality. Kane[2] provided a comprehensive discussion of capability indices which includes estimation procedures, sampling properties, and application strategies in a manufacturing environment, these indices measure larger-the-better-type quality characteristics. Hong et al.[3] developed a maximum likelihood estimate (MLE) with Pareto distribution. Hsu et al.[4] employed fuzzy inference to evaluate the lifetime performance index (LPI) when sample data from the Pareto model with censored information was imprecise, based on the right type II censored (TIIC) samples. Hong et al.[5] used a TIIC sample, created an MLE of C_L and developed a confidence interval (CI) for the LPI of Pareto distribution. Lee et al.[6] constructed a uniform minimum variance unbiased estimator (UMVUE) of C_L based on the TIIC sample under the assumption of an exponential lifetime, then the UMVUE of the LPI is utilized to develop a hypothesis testing procedure in the condition of known L.

Lee[7] used the max P-value method to select the optimum value of the shape parameter β of Weibull distribution, then, he constructed the MLE of C_L based on progressive type II censoring (PTIIC) samples from Weibull distribution, Further, a hypothesis testing procedure was developed for known L, Hsu et al.[8] offered an approach for evaluating LPI using fuzzy inference for Pareto distribution. Hong et al.[9] obtained the MLE of C_L under progressive first failure censored (PFFC) samples from Weibull distribution and developed a hypothesis testing procedure about C_L. Ahmadi et al.[10] obtained the MLE of C_L for Weibull distribution on the basis of the PFFC data. This estimate was used for developing the CI for C_L. Wu et al.[11] estimated C_L under the Burr XII distribution with the upper record values.

Ahmadi et al.[12] investigated statistical inference for C_L based on generalized order statistics (GOS). Various point and interval estimators for the parameter C_L were obtained and optimal critical regions for the hypothesis testing problems concerning C_L were proposed. Recently, Ahmadi et al.[13] assumed that the lifetimes of products are independent two-parameter exponential distribution, with a known L, and estimated C_L based on GOS. Ahmadi and Doostparast[14] calculated C_L using PFFC samples based on Pareto distribution with a known scale parameter. Hassan et al.[15] considered a MLE of C_L under Burr Type III distribution based on PTIIC. Shaabani and Jafari[16] provided an inference study on LPI of gamma distribution through point and interval estimation. Wu et al.[17] studied the experimental design for LPI of Rayleigh distribution under progressive type I interval censoring.

The concept of GOS presented by Kamps[18] enables a common approach to structural similarities and analogies. Well-known results can be subsumed, generalized, and integrated within a general framework. Several schemes of censoring can be described in terms of order statistics and can be presented by GOS.

The random variables:

$X(1,n,\tilde{n},k), X(2,n,\tilde{n},k), \ldots, X(n,n,\tilde{n},k)$ are called GOS arising from distribution function $F(x)$ with density function $f(x)$, the joint probability density function (PDF) of the above quantities is given by:

$$f_{X(1,n,\tilde{n},k),\ldots,X(n,n,\tilde{n},k)}(x_1,\ldots,x_n) = k\left(\prod_{r=1}^{n-1} \gamma_r\right)\left(\prod_{i=1}^{n-1} (1-F(x_i))^{n_i} f(x_i)\right)(1-F(x_n))^{k-1} f(x_n) \quad (2)$$

$$F^{-1}(0) < x_1 < \ldots < x_n < F^{-1}(1)$$

where $\gamma_r = k + (n-r) + N_r \geq 1$ for all $r \in \{1,2,..,n-1\}$, $N_r = \sum_{i=r}^{n-1} n_i$ with $n \in N$, $n \geq 2$, $k \geq 0$ and $\tilde{n} = (n_1, n_2, \ldots, n_{n-1}) \in \mathbb{R}^{n-1}$.

The GOS contains the following schemes:

1) Ordinary order statistics in case of $n_i = 0, i = 1,2,\ldots,n-1$ and $k = 1$.
2) GOS is reduced to usual record values from a sequence of independent and identically distributed (iid) random variables in case of $n_1 = n_2 = \ldots = n_{n-1} = -1$, and $k = 1$.
3) GOS is reduced to the first m h-record values from a sequence of iid random variables in case of $n_1 = n_2 = \ldots = n_{n-1} = -1$, and $k = h$.
4) PTIIC order statistics in case of $n_i = R_i$, $i = 1,\ldots,n-1$, $k = R_n + 1$ and $n = m$.
5) PFFC order statistics if $n_i = k(R_i + 1) - 1$ for $i = 1,\ldots,n-1$ and $k = h(R_n + 1)$.

Our aim in this research is to use data transformation to examine hypothesis testing techniques with the UMVUE of C_L and use it to judge the efficiency of any product. In this paper, we assume the lifetimes of the product follow the Pareto distribution, which is a useful model in lifetime data, and we develop Bayesian and non-Bayesian statistical inference for C_L using GOS. Then, we develop a testing procedure and the power function of the test under both Bayesian and non-Bayesian approaches, using it to find $(1-\alpha)$ one-sided CI for C_L to determine whether LPI meets the required level or not. Also, a simulation study is presented depending on two special cases of GOS, which are the PTIIC and PFFC schemes. Finally, real data are employed to assess the statistical performances of the MLEs and Bayesian estimates (BEs) for C_L of the Pareto distribution.

The next sections of the paper are organized as follows: in section 2 some properties of C_L and the connection between it and the conforming rate were given. Section 3 provides an estimation of C_L and finds its UMVUE. Section 4 gives a statistical test technique for C_L using a non-Bayesian approach and finds the power function of the given test. Section 5 investigates the Bayesian estimator of C_L under symmetric (squared error (SE)) and asymmetric (linear exponential (LINEX)) loss functions. Section 6 gives a statistical test technique for C_L in the Bayesian approach and gives the power function of the test. Section 7 provides a simulation study and real data is examined. Finally, some concluding remarks are provided in section 8.

2. The LPI and the conforming rate

Assuming the lifetime of a product follows Pareto distribution. Let X have Pareto distribution with PDF

$$f_X(x;\theta) = \theta x^{-(\theta+1)}, x \geq 1, \theta > 0 \tag{3}$$

Using the transformation $Y = \ln(X)$, then the distribution of Y is an exponential distribution, with the following PDF and the cumulative distribution function (CDF)

$$f_Y(y;\theta) = \theta e^{-\theta y}, y > 0, \theta > 0 \tag{4}$$

and,

$$F_Y(y;\theta) = 1 - e^{-\theta y}, y > 0, \theta > 0 \tag{5}$$

A longer lifetime equates to higher financial results. As a result, a lifetime is a form of quality attribute where the longer the better. Because the logarithmic transformation $Y = \ln(X)$ is one-to-one and strictly increasing, then the transformed data set of Y and the original data set of X have the same effect when evaluating lifetime performance, moreover the calculations became easier. There are various properties, including the following:

- C_L can be written as

$$C_L = \frac{\mu - L}{\sigma} = \frac{\frac{1}{\theta} - L}{\frac{1}{\theta}} = 1 - \theta L, C_L < 1 \tag{6}$$

where L is the lower specification limit.

- The failure rate function $r(y;\theta)$ is defined by:

$$r(y;\theta) = \frac{\theta e^{-\theta y}}{1 - (1 - e^{-\theta y})} = \theta \tag{7}$$

If the lifetime of the product exceeds the lower specification limit, then the product is labeled as a conforming product. Otherwise, the product is labeled as a non-conforming product. The conforming rate is defined as:

$$P_r = P(Y \geq L) = \int_L^\infty f(y)\mathrm{d}y = \int_L^\infty \theta e^{-\theta y}\mathrm{d}y = e^{-\theta L} = e^{C_L - 1}, -\infty < C_L < 1 \tag{8}$$

where Y is the lifetime variable with PDF $f(y)$ and CDF $F(y)$. In **Table 1** we give some values of P_r for some values of C_L.

Table 1 gives different C_L values and the corresponding P_r values, and shows that there is a strictly increasing relationship between conforming rate P_r and the LPI. This relationship is also graphically depicted in **Figure 1**. For any unlisted values of C_L interpolation can be used to get their corresponding P_r values. It is obvious that the relation between C_L and P_r is one-to-one, so C_L is an effective tool to estimate P_r.

Table 1. LPI versus conforming rate.

C_L	P_r	C_L	P_r	C_L	P_r
−∞	0	0.06	0.3906	0.54	0.6313
−4	0.0067	0.08	0.3985	0.56	0.644
−3.8	0.0082	0.1	0.4066	0.58	0.657
−3.6	0.0101	0.12	0.4148	0.6	0.6703
−3.4	0.0123	0.14	0.4232	0.62	0.6839
−3.2	0.015	0.16	0.4317	0.64	0.6977
−3	0.0183	0.18	0.4404	0.66	0.7118
−2.8	0.0224	0.2	0.4493	0.68	0.7261
−2.6	0.0273	0.22	0.4584	0.7	0.7408
−2.4	0.0334	0.24	0.4677	0.72	0.7558
−2.2	0.0408	0.26	0.4771	0.74	0.7711
−2	0.0498	0.28	0.4868	0.76	0.7866
−1.8	0.0608	0.3	0.4966	0.78	0.8025
−1.6	0.0743	0.32	0.5066	0.8	0.8187
−1.4	0.0907	0.34	0.5169	0.82	0.8353
−1.2	0.1108	0.36	0.5273	0.84	0.8521
−1	0.1353	0.38	0.5379	0.86	0.8694
−0.8	0.1653	0.4	0.5488	0.88	0.8869
−0.6	0.2019	0.42	0.5599	0.9	0.9048
−0.4	0.2466	0.44	0.5712	0.92	0.9231
−0.2	0.3012	0.46	0.5827	0.94	0.9418
0	0.3679	0.48	0.5945	0.96	0.9608
0.02	0.3753	0.5	0.6065	0.98	0.9802
0.04	0.3829	0.52	0.6188	1	1

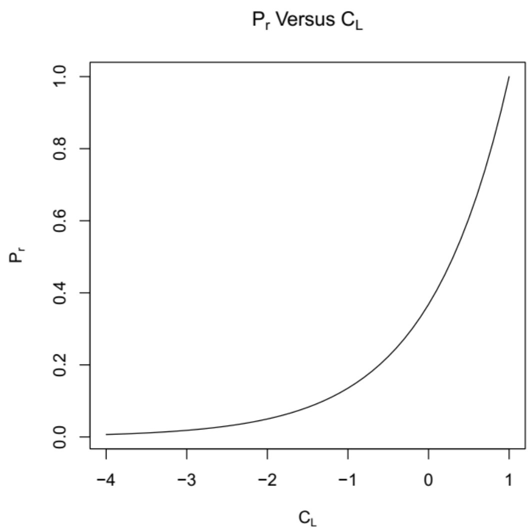

Figure 1. Conforming rate P_r versus LPI C_L.

3. UMVUE of LPI

Our aim in this section is to get UMVUE of LPI. Let $\underline{y} = (y(1,n,\tilde{n},k),\ldots,y(n,n,\tilde{n},k))$ be the GOS data from exponential (θ) distribution, and $y_i = y(i,n,\tilde{n},k)$, $i = 1,2,\ldots,n$, then by substituting Equations (4) and (5) in Equation (2) the likelihood function of Y will be:

$$L(\theta|\underline{y}) = k(\prod_{r=1}^{n-1} \gamma_r)(\theta^n)e^{-\theta(\sum_{i=1}^{n-1} y_i(n_i+1)+ky_n)}. \tag{9}$$

Putting $\gamma_n = k$, let the spacing statistics $W_i = y_i - y_{i-1}$, $i = 1,2,\ldots,n$, where $y_0 = 0$, and it is easily to show that $\sum_{i=1}^{n} \gamma_i W_i = \sum_{i=1}^{n-1} y_i(n_i + 1) + ky_n$. As a result, Equation (9) can be written as follows:

$$L(\theta|\underline{y}) = k(\prod_{r=1}^{n-1} \gamma_r)(\theta^n)e^{-\theta(\sum_{i=1}^{n} \gamma_i W_i)}. \tag{10}$$

This joint PDF belongs to the exponential family of distributions, from Theorem (5.6) of Lehmann[19] and Theorem (7.5.2) of Hogg et al.[20], we deduce that $\sum_{i=1}^{n} \gamma_i W_i$ is a complete and sufficient statistic for θ.

According to Ahmadi et al. and Ahsanullah[12,21] the random variable $2\theta\sum_{i=1}^{n} \gamma_i W_i$ follows χ^2 distribution with $2n$ degrees of freedom, written χ^2_{2n}. Furthermore, we demonstrate that the estimator $\widehat{C_L} = 1-(n-1)L(\sum_{i=1}^{n} \gamma_i W_i)^{-1}$ is unbiased estimator of C_L.

$$E(\widehat{C_L}) = E\left(1-(n-1)L\left(\sum_{i=1}^{n} \gamma_i W_i\right)^{-1}\right) = E\left(1-(n-1)(2\theta)L\frac{1}{2\theta\sum_{i=1}^{n} \gamma_i W_i}\right)$$
$$= 1-(n-1)(2\theta)LE\left(\frac{1}{2\theta\sum_{i=1}^{n} \gamma_i W_i}\right).$$

It is clear that the term $\frac{1}{2\theta\sum_{i=1}^{n} \gamma_i W_i}$ has an inverted χ^2 with mean $\frac{1}{2(n-1)}$, so the previous equation takes the following form:

$$E(\widehat{C_L}) = 1-(n-1)(2\theta)L\frac{1}{2(n-1)} = 1-\theta L. \tag{11}$$

Equation (11) proved that $\widehat{C_L}$ is unbiased estimator of C_L, according to Theorem (7.4.1) in Hogg et al.[20] and corollary (1.12) in Lehmann and Casella[22], then $\widehat{C_L}$ is UMVUE of C_L.

4. Testing procedure for the LPI and the power function of the test

In this section, we study the testing procedures for the LPI and the power function of the test.

4.1. Testing procedure for the LPI

Create a statistical test technique to determine whether the LPI complies with the necessary standard. Assuming that the required value of LPI is more than C where C is the target value. Our aim is to test the null hypothesis.

$$\mathrm{H}_0: C_L \leq C. \tag{12}$$

The product is unreliable.

Against the alternative hypothesis.

$$\mathrm{H}_1: C_L > C. \tag{13}$$

The product is reliable.

The UMVUE $\widehat{C_L}$ of C_L is used as the test statistics, the rejection region can be expressed as $\{\widehat{C_L}|\widehat{C_L} > C_0\}$. With specified significance level α, the critical value C_0 can be calculated as follows:

$$P(\widehat{C_L} > C_0 | C_L = C) = \alpha$$

$$\Rightarrow P(1 - \frac{(n-1)L}{\sum_{i=1}^{n} \gamma_i W_i} > C_0 | 1 - \theta L = C) = \alpha.$$

Putting $W^* = \sum_{i=1}^{n} \gamma_i W_i$.

$$\Rightarrow P\left(1 - \frac{(n-1)L}{W^*} > C_0 | \theta = \frac{1-C}{L}\right) = \alpha$$

$$\Rightarrow P\left(1 - \frac{(n-1)L}{W^*} \leq C_0 | \theta = \frac{1-C}{L}\right) = 1 - \alpha$$

$$\Rightarrow P\left(1 - \frac{2\theta(n-1)L}{2\theta W^*} \leq C_0 | \theta = \frac{1-C}{L}\right) = 1 - \alpha \tag{14}$$

$$\Rightarrow P\left(2\theta W^* \leq \frac{2(1-C)(n-1)}{1-C_0}\right) = 1 - \alpha.$$

As we know $2\theta W^* \sim \chi^2_{(2n)}$ and from Equation (14) using the inverse of $\chi^2_{(2n)}$ $(INVCHI)$, then $INVCHI(1-\alpha, 2n)$ is the lower $(1-\alpha)$ percentile of $\chi^2_{(2n)}$, so

$$\frac{2(1-C)(n-1)}{1-C_0} = INVCHI(1-\alpha, 2n).$$

Then, the following critical value can be derived:

$$C_0 = 1 - \frac{2(1-C)(n-1)}{INVCHI(1-\alpha, 2n)}, \tag{15}$$

where C is the target value, α is the specified significance and n is the observed number of GOS samples.

4.2. The power function of the test

The power of the statistical test is the probability of correctly rejecting the false null hypothesis. Applying the null hypothesis Equation (12) against the alternative hypothesis Equation (13), we get a size α test with the rejection region $\{\widehat{C_L} > C_0\}$. The power $P(C_1)$ of the test at $C_L = C_1(> C)$ is then

$$P(C_1) = P(\widehat{C_L} > C_0)$$

$$= P\left(\widehat{C_L} > 1 - \frac{2(1-C)(n-1)}{INVCHI(1-\alpha, 2n)} | C_L = C_1\right)$$

$$= P\left(1 - \frac{(n-1)L}{W^*} > 1 - \frac{2(1-C)(n-1)}{INVCHI(1-\alpha, 2n)} | \theta = \frac{1-C_1}{L}\right) \tag{16}$$

$$= P\left(2\theta W^* > \frac{\theta INVCHI(1-\alpha, 2n)L}{1-C} | \theta = \frac{1-C_1}{L}\right)$$

$$= P\left(2\theta W^* > \frac{(1-C_1)INVCHI(1-\alpha, 2n)}{1-C}\right).$$

To find $(1-\alpha)$ one-sided CI for C_L using the pivotal quantity $2\theta W^*$, where $2\theta W^* \sim \chi^2_{(2n)}$, employing $INVCHI(1-\alpha, 2n)$ function which represents the lower $(1-\alpha)$ percentile of $\chi^2_{(2n)}$.

$$P(2\theta W^* \leq INVCHI(1-\alpha, 2n)) = 1 - \alpha$$

$$\Rightarrow P\left(\frac{2(n-1)(1-C_L)}{1-\widehat{C_L}} \leq INVCHI(1-\alpha, 2n)\right) = 1 - \alpha \tag{17}$$

$$\Rightarrow P\left(C_L \geq 1 - \frac{(1-\widehat{C_L})INVCHI(1-\alpha, 2n)}{2(n-1)}\right) = 1 - \alpha,$$

where $C_L = 1 - \theta L$ and $\widehat{C_L} = 1 - \frac{(n-1)L}{W^*}$.

From Equation (17), the level $(1-\alpha)$ one sided CI lower bound ($\underline{LB}$) for C_L is:

$$\underline{LB} = 1 - \frac{(1-\widehat{C_L})INVCHI(1-\alpha,2n)}{2(n-1)}. \tag{18}$$

Hence, the decision rule for the test is "if $c \notin [\underline{LB},\infty)$", then the LPI meets the required level.

5. Bayesian inference for C_L

Bayesian inference has received great attention because of its validity in inference more than the traditional method that depends on frequencies. It treats parameter as a random variable and it combines its prior distribution denoted by $\pi(\theta)$ with the information contained in the sample denoted by $L(\underline{y}|\theta)$ to get the posterior distribution of the parameter.

5.1. The BE of C_L under symmetric loss function

Let θ have gamma prior distribution with hyper-parameters a and b,

$$\pi(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}, a>0, b>0. \tag{19}$$

Using the Bayes' theorem, the posterior distribution of θ comes from Equations (10) and (19),

$$\pi(\theta|\underline{y}) \propto \theta^{a+n-1}e^{-\theta(\sum_{i=1}^{n}\gamma_i W_i + b)}, \tag{20}$$

then $\pi(\theta|\underline{y}) \sim \Gamma(a+n, \sum_{i=1}^{n}\gamma_i W_i + b)$. Further, the BE of θ under SE loss function $L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$ is the mean of the posterior density which is

$$\hat{\theta}_{SE} = \frac{a+n}{\sum_{i=1}^{n}\gamma_i W_i + b}. \tag{21}$$

From Equations (6) and (21), we have

$$\hat{C}_{L_{SE}} = 1 - \frac{(a+n)L}{\sum_{i=1}^{n}\gamma_i W_i + b}.$$

Let $W^{**} = \sum_{i=1}^{n}\gamma_i W_i + b$, then Equation (20) can be written as:

$$\pi(\theta|\underline{y}) = \frac{(W^{**})^{a+n}}{\Gamma(a+n)}\theta^{a+n-1}e^{-\theta W^{**}}. \tag{22}$$

Using change of variables (see Casella and Berger[23]), let $Z = 2\theta W^{**}$, then the PDF of Z is given by:

$$\pi(\theta|\underline{y}) = \frac{(W^{**})^{a+n}}{\Gamma(a+n)}\theta^{a+n-1}e^{-\theta W^{**}}. \tag{23}$$

So, $2\theta W^{**} \sim \chi^2_{2(a+n)}$ provided that "*a*" is a positive integer and $\hat{C}_{L_{SE}} = 1 - \left(\frac{(a+n)L}{W^{**}}\right)$.

Remark 1. *The expectation of* $\hat{C}_{L_{SE}}$ *is*

$$E(\hat{C}_{L_{SE}}) = 1 - 2(a+n)\theta L E\left(\frac{1}{2\theta W^{**}}\right) = 1 - \frac{(a+n)\theta L}{(a+n)-1}. \tag{24}$$

From Equation (24), we conclude that the BE $(\hat{C}_{L_{SE}})$ is not unbiased estimator of C_L. When $n \Rightarrow \infty$, $E(\hat{C}_{L_{SE}}) \Rightarrow C_L$, so the BE $\hat{C}_{L_{SE}}$ is asymptotically unbiased estimator.

5.2. The BE C_L under asymmetric loss function

Varian[24] introduced the LINEX as an asymmetric loss function defined by:

$$L(\hat{\theta},\theta) \propto e^{\tau(\hat{\theta}-\theta)} - \tau(\hat{\theta}-\theta) - 1,$$

where $\tau \neq 0$ is a known shape parameter, the BE for θ and C_L under LINEX loss function is, respectively, given by:

$$\hat{\theta}_{LX} = \frac{-1}{\tau}\ln E(e^{-\tau\theta}|\underline{y}), \tag{25}$$

and,

$$\hat{C}_{L_{LX}} = \frac{-1}{\tau}\ln E(e^{-\tau(1-\theta L)}|\underline{y}). \tag{26}$$

Using Equation (22), then $E(e^{-\tau(1-\theta L)}|\underline{y})$ is determined as below:

$$E(e^{-\tau(1-\theta L)}|\underline{y}) = e^{-\tau}E(e^{\tau\theta L}|\underline{y})$$

$$= e^{-\tau}\int_0^\infty \frac{(W^{**})^{a+n}}{\Gamma(a+n)}\theta^{a+n-1}e^{-(\theta W^{**}-\tau\theta L)}\mathrm{d}\theta$$

$$= e^{-\tau}\frac{(W^{**})^{a+n}}{(W^{**}-\tau L)^{a+n}}$$

$$= e^{-\tau}\left(1-\frac{\tau L}{W^{**}}\right)^{-(a+n)}.$$

Then by inserting the previous equation in Equation (26), leads to

$$\hat{C}_{L_{LX}} = \frac{-1}{\tau}\ln\left(e^{-\tau}\left(1-\frac{\tau L}{W^{**}}\right)^{-(a+n)}\right) = 1+\frac{a+n}{\tau}\ln\left(1-\frac{\tau L}{W^{**}}\right).$$

One can show that $\hat{C}_{L_{LX}}$ converges to $\hat{C}_{L_{SE}}$ as $\tau \longrightarrow 0$.

6. Testing procedure for the LPI in Bayesian approach and the power function of the test

In this section, we study the testing procedure for the LPI in the Bayesian approach and the power function of the test.

6.1. Testing procedure for the LPI in Bayesian approach

Here, we create a statistical test technique to determine whether the LPI complies with the necessary standards. Assuming that the required value of LPI is more than C where C is the target value. Our aim is to test the null hypothesis Equation (12) against the alternative hypothesis Equation (13). The $\hat{C}_{L_{SE}}$ of C_L is used as the test statistics, and the rejection region can be expressed as $\{\hat{C}_{L_{SE}} > C_0\}$. With specified significance level α, the critical value C_0 can be calculated as follows:

$$P(\hat{C}_{L_{SE}} > C_0|C_L = C) = \alpha$$

$$\Rightarrow P\left(1-\left(\frac{(a+n)L}{W^{**}}\right) > C_0|1-\theta L = C\right) = \alpha$$

$$\Rightarrow P(1-\frac{(a+n)L}{W^{**}} > C_0|\theta = \frac{1-C}{L}) = \alpha$$

$$\Rightarrow P(1-\frac{(a+n)L}{W^{**}} \le C_0|\theta = \frac{1-C}{L}) = 1-\alpha \tag{27}$$

$$\Rightarrow P(1-\frac{2\theta(a+n)L}{2\theta W^{**}} \le C_0|\theta = \frac{1-C}{L}) = 1-\alpha$$

$$\Rightarrow P(2\theta W^{**} \le \frac{2(1-C)(a+n)}{1-C_0}) = 1-\alpha.$$

As we know $2\theta W^{**} \sim \chi^2_{(2(n+a))}$ and from Equation (27) using $INVCHI(1-\alpha, 2(a+n))$ function that is the lower $(1-\alpha)$ percentile of $\chi^2_{(2(a+n))}$ so,

$$\frac{2(1-C)(a+n)}{1-C_0} = INVCHI(1-\alpha, 2(n+a)).$$

Then, the following critical value can be derived

$$C_0 = 1 - \frac{2(1-C)(a+n)}{INVCHI(1-\alpha, 2(a+n))}. \tag{28}$$

6.2. The power function of the test

The power of the statistical test is the probability of correctly rejecting the false null hypothesis. Applying the null hypothesis Equation (12) against the alternative hypothesis Equation (13), one gets a size α test with the rejection region $\{\hat{C}_{L_{SE}} > C_0\}$. The power $P(C_1)$ of the test at $C_L = c_1(> c)$ is then

$$\begin{aligned} P(C_1) &= P(\hat{C}_{L_{SE}} > C_0) \\ &= P\left(\hat{C}_{L_{SE}} > 1 - \frac{2(1-C)(a+n)}{INVCHI(1-\alpha, 2(a+n))} | C_L = c_1\right) \\ &= P\left(1 - \frac{(a+n)L}{W^{**}} > 1 - \frac{2(1-C)(a+n)}{INVCHI(1-\alpha, 2(a+n))} | \theta = \frac{1-c_1}{L}\right) \\ &= P\left(2\theta W^{**} > \frac{\theta INVCHI(1-\alpha, 2n)L}{1-c} | \theta = \frac{1-c_1}{L}\right) \\ &= P\left(2\theta W^{**} > \frac{(1-c_1) INVCHI(1-\alpha, 2(a+n))}{1-c}\right). \end{aligned} \tag{29}$$

The level $(1-\alpha)$ one-sided credible interval for C_L can be done as follows:

With pivotal quantity $2\theta W^{**}$, where $2\theta W^{**} \sim \chi^2_{(2(a+n))}$, and $INVCHI(1-\alpha, 2(n+a))$ function which represents the lower $(1-\alpha)$ percentile of $\chi^2_{(2(a+n))}$.

$$\begin{aligned} & P(2\theta W^{**} \le INVCHI(1-\alpha, 2(a+n))) = 1-\alpha \\ & \Rightarrow P\left(\frac{2(a+n)(1-C_L)}{1-\hat{C}_{L_{SE}}} \le INVCHI(1-\alpha, 2(a+n))\right) = 1-\alpha, \\ & \Rightarrow P\left(C_L \ge 1 - \frac{(1-\hat{C}_{L_{SE}}) INVCHI(1-\alpha, 2(a+n))}{2(a+n)}\right) = 1-\alpha, \end{aligned} \tag{30}$$

where $C_L = 1 - \theta L$ and $\hat{C}_{L_{SE}} = 1 - \frac{(a+n)L}{W^{**}}$, from Equation (30) the level $(1-\alpha)$ one sided credible interval $\underline{LB}_{Bs}$ for C_L lower credible bound for C_L is:

$$\underline{LB} = 1 - \frac{(1-\hat{C}_{L_{Bs}}) INVCHI(1-\alpha, 2(a+n))}{2(a+n)}. \tag{31}$$

Now the decision rule for the test is “if $c \notin [\underline{LB}, \infty)$” then the lifetime index meets the required level.

7. Simulation study and real data analysis

In this section, the purpose is to analyze the performance of the estimation methods presented in the sections above. For illustrative purposes, a real data set is used. Furthermore, a simulation study is used to investigate the behavior of the proposed methods and to test the statistical performances of the estimates given the PTIIC and PFFC as two censoring schemes. Calculations have been performed using the R-statistical programming language, calculations are done by utilizing the bbmle package.

7.1. Simulation study

Here, the simulation study is used to analyze the performance of the proposed estimation methods under the PTIIC and PFFC schemes.

7.1.1. Part I: PTIIC scheme

To analyze the performance of estimation methods, including ML and Bayesian, a Monte Carlo simulation study is employed, under the PTIIC scheme for Pareto distribution. 1000 observations are generated from Pareto distribution then apply the transformation ($Y = \ln X$) for the MLEs under the following assumptions:

1) True parameter value of Pareto distribution is selected as: $\theta = 0.5$, and 1.5.
2) Lower specification limit is assumed to be $L = 0.25$ and 1.
3) Sample sizes of $n = 40$ and 80.
4) Removed items R_i are assumed at different sample sizes n and number of stages m.
5) **Table 2** provides the selected censoring schemes, namely scheme 1 (S_1), scheme 2 (S_2), scheme 3 (S_3), and scheme 4 (S_4) at different values of n and m.

Table 2. Numerous patterns for removing items from life test at different number of stages.

n	***m***	**Censoring schemes**			
		S_1	S_2	S_3	S_4
40	20	$(20,\ 0^{*19})$	$(10,\ 0^{*18}, 10)$	$(0^{*9}, 10, 10,\ 0^{*9})$	$(0^{*19}, 20)$
	30	$(10,\ 0^{*29})$	$(5,\ 0^{*28}, 5)$	$(0^{*14}, 5, 5,\ 0^{*14})$	$(0^{*29}, 10)$
80	40	$(40,\ 0^{*39})$	$(20,\ 0^{*38}, 20)$	$(0^{*19}, 20, 20,\ 0^{*19})$	$(0^{*39}, 40)$
	60	$(20,\ 0^{*59})$	$(10,\ 0^{*58}, 10)$	$(0^{*29}, 10, 10,\ 0^{*29})$	$(0^{*59}, 20)$

Here, $(5^{*3},\ 0)$, for example, means that the censoring scheme employed is (5, 5, 5, 0).

The MLEs of $\widehat{C_L}$ are produced based on the generated data. To solve the numerical equations to get MLEs, we take the true parameter values as initial values. For the Bayesian method, we compute the BEs using informative priors (IPs) for the gamma distribution with parameters $a = 0.5$, and $b = 1.5$. Such values of IPs are plugged-in to evaluate the required estimates. The BEs were computed under two loss functions: SE and LINEX ($\tau = -0.5, 0.5$).

All the average estimates are reported in **Tables 3** and **4** for $\theta = 0.5$, $L = 0.25$ and $L = 1$, respectively, and in **Tables 5** and **6** for $\theta = 1.5$, $L = 0.25$ and $L = 1$, respectively. Further, the first column denotes the average estimates (Avg.), and in the second column, related mean squared errors (MSEs).

Table 3. Average estimated values and MSEs of the ML and Bayesian for different PTIIC schemes for Pareto distribution with $\theta = 0.5$, $L = 0.25$ and true value of $C_L = 0.875$.

(*n*, *m*)	**Sch.**	C_L							
		MLE		**BE:SE**		**BE:LINEX**			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
(40, 20)	S_1	0.8682	0.001	0.8186	0.0034	0.8198	0.0033	0.819	0.0034
	S_2	0.8676	0.001	0.8817	0.0003	0.8822	0.0003	0.8819	0.0003
	S_3	0.8667	0.001	0.866	0.0003	0.8667	0.0003	0.8662	0.0003
	S_4	0.8675	0.001	0.893	0.0005	0.8934	0.0005	0.8932	0.0005
(40, 30)	S_1	0.8704	0.0007	0.8031	0.0053	0.804	0.0052	0.8034	0.0053
	S_2	0.8711	0.0006	0.855	0.0006	0.8555	0.0005	0.8551	0.0006
	S_3	0.8712	0.0006	0.8393	0.0014	0.8399	0.0014	0.8395	0.0014

Table 3. (*Continued*).

(*n*, *m*)	**Sch.**	C_L							
		MLE		**BE:SE**		**BE:LINEX**			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
	S_4	0.8697	0.0007	0.8712	0.0002	0.8716	0.0002	0.8713	0.0002
(80, 40)	S_1	0.8724	0.0004	0.7975	0.0061	0.7982	0.006	0.7977	0.0061
	S_2	0.872	0.0005	0.8817	0.0002	0.882	0.0002	0.8818	0.0002
	S_3	0.8721	0.0004	0.8585	0.0004	0.8589	0.0004	0.8586	0.0004
	S_4	0.8719	0.0004	0.8945	0.0005	0.8947	0.0005	0.8945	0.0005
(80, 60)	S_1	0.8731	0.0003	0.7852	0.0081	0.7858	0.008	0.7854	0.0081
	S_2	0.8727	0.0003	0.8519	0.0006	0.8522	0.0006	0.852	0.0006
	S_3	0.8732	0.0003	0.8288	0.0022	0.8291	0.0022	0.8289	0.0022
	S_4	0.8727	0.0003	0.871	0.0001	0.8712	0.0001	0.8711	0.0001

Table 4. Average estimated values and MSEs of the ML and Bayesian for different PTIIC schemes for Pareto distribution with $\theta = 0.5$, $L = 1$ and true value of $C_L = 0.5$.

(*n*, *m*)	**Sch.**	C_L							
		MLE		**BE:SE**		**BE:LINEX**			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
(40, 20)	S_1	0.4748	0.0158	0.2656	0.0596	0.285	0.0504	0.2722	0.0564
	S_2	0.4726	0.0149	0.5241	0.0039	0.5324	0.0041	0.5269	0.0039
	S_3	0.4724	0.0165	0.4616	0.0055	0.4722	0.0045	0.4652	0.0051
	S_4	0.4809	0.0147	0.5733	0.0086	0.58	0.0094	0.5756	0.0089
(40, 30)	S_1	0.4862	0.0098	0.2041	0.0903	0.2194	0.0812	0.2093	0.0872
	S_2	0.478	0.0095	0.4137	0.0098	0.4221	0.0083	0.4165	0.0093
	S_3	0.482	0.0095	0.35	0.0249	0.3603	0.0218	0.3534	0.0239
	S_4	0.4811	0.01	0.4841	0.0028	0.4907	0.0025	0.4863	0.0027
(80, 40)	S_1	0.4819	0.0072	0.1794	0.1046	0.1917	0.0968	0.1835	0.1019
	S_2	0.4833	0.0074	0.5216	0.0022	0.5259	0.0024	0.5231	0.0023
	S_3	0.4886	0.0074	0.4317	0.007	0.4377	0.0061	0.4338	0.0067
	S_4	0.4867	0.0073	0.5757	0.0074	0.5791	0.0079	0.5768	0.0076
(80, 60)	S_1	0.4922	0.0047	0.1345	0.1347	0.1437	0.128	0.1376	0.1325
	S_2	0.4928	0.0044	0.4051	0.0103	0.4095	0.0095	0.4066	0.01
	S_3	0.4917	0.0043	0.3108	0.037	0.3167	0.0348	0.3128	0.0362
	S_4	0.4923	0.0049	0.484	0.0016	0.4873	0.0015	0.4851	0.0016

Table 5. Average estimated values and MSEs of the ML and Bayesian for different PTIIC schemes for Pareto distribution with $\theta = 1.5$, $L = 0.25$ and the true value of $C_L = 0.625$.

(*n*, *m*)	**Sch.**	C_L							
		MLE		**BE:SE**		**BE:LINEX**			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
(40, 20)	S_1	0.6061	0.0089	0.7778	0.0234	0.7796	0.024	0.7784	0.0236
	S_2	0.6045	0.0084	0.8181	0.0374	0.8193	0.0379	0.8185	0.0376
	S_3	0.6043	0.0093	0.8066	0.0331	0.8079	0.0336	0.807	0.0333
	S_4	0.6107	0.0083	0.8283	0.0415	0.8294	0.0419	0.8287	0.0416
(40, 30)	S_1	0.6146	0.0055	0.7702	0.0211	0.7715	0.0215	0.7707	0.0213
	S_2	0.6085	0.0054	0.7982	0.0301	0.7992	0.0304	0.7985	0.0302
	S_3	0.6115	0.0053	0.7886	0.0268	0.7897	0.0272	0.789	0.0269
	S_4	0.6108	0.0057	0.8102	0.0344	0.8111	0.0347	0.8105	0.0345
(80, 40)	S_1	0.6114	0.0041	0.7674	0.0203	0.7684	0.0206	0.7677	0.0204
	S_2	0.6125	0.0041	0.8171	0.037	0.8178	0.0372	0.8174	0.0371
	S_3	0.6165	0.0041	0.801	0.0311	0.8018	0.0313	0.8013	0.0311
	S_4	0.615	0.0041	0.8284	0.0415	0.8289	0.0417	0.8286	0.0415
(80, 60)	S_1	0.6191	0.0027	0.7626	0.0189	0.7633	0.0191	0.7628	0.019
	S_2	0.6196	0.0025	0.7966	0.0295	0.7971	0.0296	0.7967	0.0295
	S_3	0.6187	0.0024	0.7829	0.025	0.7835	0.0252	0.7831	0.025
	S_4	0.6192	0.0028	0.8099	0.0342	0.8103	0.0344	0.81	0.0343

Table 6. Average estimated values and MSEs of the ML and Bayesian for different PTIIC schemes for Pareto distribution with $\theta = 1.5$, $L = 1$ and true value of $C_L = -0.5$.

(*n*, *m*)	**Sch.**	C_L							
		MLE		**BE:SE**		**BE:LINEX**			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
(40, 20)	S_1	−0.5755	0.1423	0.0963	0.3569	0.1254	0.3922	0.1063	0.3688
	S_2	−0.5822	0.1339	0.2624	0.5834	0.2819	0.6133	0.269	0.5935
	S_3	−0.5827	0.1482	0.215	0.5133	0.237	0.5451	0.2225	0.524
	S_4	−0.5574	0.1323	0.3045	0.6497	0.3219	0.6778	0.3104	0.6592
(40, 30)	S_1	−0.5415	0.0885	0.0703	0.3258	0.0911	0.35	0.0774	0.3339
	S_2	−0.566	0.0855	0.1846	0.4696	0.2006	0.4918	0.19	0.4771
	S_3	−0.554	0.0854	0.1454	0.4174	0.1631	0.4404	0.1514	0.4251
	S_4	−0.5567	0.0904	0.2337	0.5396	0.2479	0.5606	0.2385	0.5467
(80, 40)	S_1	−0.5544	0.0648	0.0613	0.3154	0.0774	0.3337	0.0667	0.3215
	S_2	−0.5501	0.0662	0.2635	0.5841	0.2735	0.5993	0.2669	0.5892
	S_3	−0.5341	0.0662	0.1982	0.4886	0.21	0.5051	0.2022	0.4941
	S_4	−0.5401	0.066	0.3091	0.656	0.3179	0.6702	0.3121	0.6608
(80, 60)	S_1	−0.5234	0.0426	0.0446	0.2968	0.0558	0.3091	0.0484	0.3009
	S_2	−0.5216	0.0396	0.1821	0.4658	0.1903	0.4771	0.1849	0.4696
	S_3	−0.5251	0.0387	0.127	0.3935	0.1364	0.4053	0.1302	0.3975
	S_4	−0.5232	0.044	0.236	0.5424	0.2431	0.553	0.2384	0.5459

The findings of the simulation study are shown in **Table 3** through **Table 6**, and they include the average value, and MSE of the MLE and BE under SE, and LINEX loss functions for C_L using the chosen PTIIC schemes. The following conclusions may be drawn from these tables:

- The bias of $\widehat{C_L}$ based on ML method is smaller than its corresponding of C_L estimate based on Bayesian method.
- The bias of $\widehat{C_L}$ based on ML method decreases as n and m increase.
- The MSE based on the ML method decreases as n and m increase, which means that the MLE is consistent.
- For all tables and all schemes, the MSE based on ML method is lower than the MSE of the Bayesian method.
- The average of $\widehat{C_L}$ using the ML method is generally closer to the true value of the parameter than the average of $\widehat{C_L}$ using the Bayesian method.
- In all four tables, the average of $\widehat{C_L}$ based on ML method is closest to the true value for the S_4 scheme.

7.1.2. Part II: PFFC scheme

In this sub-section, to analyze the performance of estimation methods, including ML and Bayesian, a Monte Carlo simulation study is employed, under PFFC scheme for Pareto distribution. For the MLEs, 1000 observations are generated from the Pareto distribution then apply the transformation ($Y = \ln X$) based on the following assumptions:

1) The true parameter of Pareto distribution is given as $\theta = 0.5$ and 1.5.
2) Lower specification limit is assumed as $L = 0.25, 1$.
3) Number of groups is n, where $n = 40$ and 80.
4) Number of items in each group is k given by: $k = 3,5$. Note that: if $k = 1$, the PFFC is reduced to PTIIC.
5) Removed items R_i are similar to those for PTIIC in **Table 2**.

The MLEs of C_L is produced based on the generated data. To solve the numerical equations to get MLEs, we take the true parameter values as initial guess values. For the Bayesian method, we compute BEs using IPs with parameters $a = 0.5$ and $b = 1.5$. Such values of IPs are plugged-in to evaluate the required estimates. The Bayes estimates were computed under SE and LINEX ($\tau = -0.5, 0.5$) loss functions.

All the average estimates for methods are reported in **Tables 7** and **8** for $\theta = 0.5$, $L = 0.25$, and $L = 1$, respectively, and in **Tables 9** and **10** for $\theta = 1.5$, $L = 0.25$, and $L = 1$, respectively. Further, the first column donates Avg. estimates, and the second column is the related MSEs.

Table 7. Average estimated values and MSEs of the ML and Bayesian for different PFFC schemes for Pareto distribution with $\theta = 0.5$, $L = 0.25$ and true value of $C_L = 0.875$.

(*k*, *n*, *m*)	**Sch.**	C_L							
		MLE		**BE:SE**		**BE:LINEX**			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
(3, 40, 20)	S_1	0.8686	0.001	0.8146	0.0039	0.8158	0.0038	0.815	0.0039
	S_2	0.8699	0.001	0.8821	0.0003	0.8826	0.0003	0.8822	0.0003
	S_3	0.8694	0.0009	0.8325	0.002	0.8335	0.0019	0.8328	0.002
	S_4	0.8684	0.001	0.8934	0.0005	0.8938	0.0005	0.8935	0.0005
(3, 40, 30)	S_1	0.8709	0.0006	0.8016	0.0056	0.8026	0.0054	0.8019	0.0055
	S_2	0.871	0.0006	0.8544	0.0006	0.8549	0.0006	0.8546	0.0006

Table 7. (*Continued*).

(*k*, *n*, *m*)	**Sch.**	C_L							
		MLE		**BE:SE**		**BE:LINEX**			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
	S_3	0.872	0.0006	0.8142	0.0038	0.815	0.0037	0.8145	0.0038
	S_4	0.8699	0.0006	0.8716	0.0002	0.872	0.0002	0.8717	0.0002
(3, 80, 40)	S_1	0.8704	0.0005	0.7933	0.0068	0.7941	0.0067	0.7935	0.0067
	S_2	0.8721	0.0004	0.8812	0.0001	0.8814	0.0001	0.8812	0.0001
	S_3	0.8723	0.0005	0.8191	0.0032	0.8197	0.0032	0.8193	0.0032
	S_4	0.8722	0.0005	0.8946	0.0005	0.8948	0.0005	0.8947	0.0005
(3, 80, 60)	S_1	0.8723	0.0003	0.7834	0.0085	0.7839	0.0084	0.7835	0.0084
	S_2	0.8725	0.0003	0.8512	0.0006	0.8515	0.0006	0.8513	0.0006
	S_3	0.8729	0.0003	0.7999	0.0057	0.8004	0.0056	0.8	0.0057
	S_4	0.8732	0.0003	0.8711	0.0001	0.8713	0.0001	0.8712	0.0001
(5, 40, 20)	S_1	0.8687	0.001	0.8139	0.004	0.8151	0.0038	0.8143	0.004
	S_2	0.8682	0.0009	0.881	0.0002	0.8815	0.0002	0.8812	0.0002
	S_3	0.8681	0.001	0.8227	0.003	0.8239	0.0028	0.8231	0.0029
	S_4	0.8702	0.0009	0.8942	0.0006	0.8946	0.0006	0.8943	0.0006
(5, 40, 30)	S_1	0.8715	0.0006	0.801	0.0056	0.802	0.0055	0.8013	0.0056
	S_2	0.8695	0.0006	0.8539	0.0006	0.8544	0.0006	0.8541	0.0006
	S_3	0.8705	0.0006	0.8077	0.0047	0.8086	0.0046	0.808	0.0046
	S_4	0.8703	0.0006	0.8719	0.0002	0.8723	0.0002	0.872	0.0002
(5, 80, 40)	S_1	0.8705	0.0005	0.7933	0.0068	0.794	0.0067	0.7935	0.0067
	S_2	0.8708	0.0005	0.8803	0.0001	0.8806	0.0001	0.8804	0.0001
	S_3	0.8722	0.0005	0.8079	0.0046	0.8085	0.0045	0.8081	0.0046
	S_4	0.8717	0.0005	0.8944	0.0005	0.8946	0.0005	0.8944	0.0005
(5, 80, 60)	S_1	0.8731	0.0003	0.7835	0.0084	0.7841	0.0083	0.7837	0.0084
	S_2	0.8732	0.0003	0.8516	0.0006	0.8518	0.0006	0.8517	0.0006
	S_3	0.8729	0.0003	0.7928	0.0068	0.7933	0.0067	0.793	0.0068
	S_4	0.8731	0.0003	0.8714	0.0001	0.8716	0.0001	0.8715	0.0001

Table 8. Avg. estimated values and MSEs of the ML and Bayesian for different schemes of PFFC for Pareto distribution with $\theta = 0.5$, $L = 1$ and the true value of $C_L = 0.5$.

(*k*, *n*, *m*)	**Sch.**	C_L							
		MLE		**BE:SE**		**BE:LINEX**			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
(3, 40, 20)	S_1	0.4696	0.0176	0.2458	0.0688	0.2662	0.0584	0.2528	0.0651
	S_2	0.4731	0.0167	0.5212	0.0041	0.5296	0.0043	0.524	0.0042
	S_3	0.475	0.0158	0.3229	0.0354	0.3394	0.0294	0.3285	0.0333
	S_4	0.4768	0.0151	0.5717	0.0083	0.5784	0.0091	0.574	0.0086
(3, 40, 30)	S_1	0.4805	0.01	0.1957	0.0951	0.2114	0.0856	0.201	0.0918
	S_2	0.4854	0.0098	0.4146	0.0098	0.423	0.0083	0.4174	0.0093

Table 8. (*Continued*).

(*k, n, m*)	Sch.	C_L							
		MLE		BE:SE		BE:LINEX			
						$\tau = -0.5$		$\tau = -0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
	S_3	0.4802	0.0105	0.2487	0.0655	0.2624	0.0586	0.2534	0.0631
	S_4	0.485	0.0087	0.4864	0.0025	0.4929	0.0023	0.4886	0.0025
(3, 80, 40)	S_1	0.4883	0.0067	0.1703	0.1106	0.1829	0.1024	0.1746	0.1078
	S_2	0.4888	0.0072	0.5224	0.0022	0.5266	0.0024	0.5238	0.0023
	S_3	0.4817	0.0075	0.2695	0.0548	0.2793	0.0503	0.2728	0.0533
	S_4	0.4889	0.0072	0.5765	0.0075	0.5799	0.008	0.5776	0.0077
(3, 80, 60)	S_1	0.4888	0.0048	0.1278	0.1395	0.1372	0.1325	0.131	0.1371
	S_2	0.4875	0.0044	0.403	0.0106	0.4074	0.0098	0.4044	0.0103
	S_3	0.4909	0.0045	0.194	0.0945	0.202	0.0896	0.1967	0.0929
	S_4	0.4958	0.0044	0.4849	0.0015	0.4882	0.0013	0.486	0.0014
(5, 40, 20)	S_1	0.4696	0.0176	0.242	0.0707	0.2626	0.0601	0.249	0.067
	S_2	0.4731	0.0167	0.5205	0.0041	0.5289	0.0043	0.5233	0.0041
	S_3	0.475	0.0158	0.2847	0.0506	0.3032	0.0426	0.291	0.0478
	S_4	0.4768	0.0151	0.5717	0.0083	0.5784	0.0091	0.574	0.0086
(5, 40, 30)	S_1	0.4805	0.01	0.1943	0.0959	0.2101	0.0863	0.1997	0.0926
	S_2	0.4854	0.0098	0.4142	0.0099	0.4226	0.0084	0.4171	0.0094
	S_3	0.4802	0.0105	0.2243	0.0785	0.2389	0.0704	0.2292	0.0757
	S_4	0.485	0.0087	0.4864	0.0025	0.4929	0.0023	0.4886	0.0025
(5, 80, 40)	S_1	0.4883	0.0067	0.168	0.1121	0.1807	0.1038	0.1723	0.1093
	S_2	0.4888	0.0072	0.522	0.0022	0.5263	0.0024	0.5235	0.0023
	S_3	0.4817	0.0075	0.2258	0.0769	0.2367	0.0709	0.2295	0.0748
	S_4	0.4889	0.0072	0.5765	0.0075	0.5799	0.008	0.5776	0.0077
(5, 80, 60)	S_1	0.4888	0.0048	0.127	0.1401	0.1364	0.1331	0.1301	0.1377
	S_2	0.4875	0.0044	0.4028	0.0107	0.4072	0.0098	0.4042	0.0104
	S_3	0.4909	0.0045	0.1656	0.1127	0.1742	0.107	0.1685	0.1107
	S_4	0.4958	0.0044	0.4849	0.0015	0.4882	0.0013	0.486	0.0014

Table 9. Avg. estimated values and MSEs of the ML and Bayesian for different PFFC schemes for Pareto distribution with $\theta = 1.5$, $L = 0.25$ and true value of $C_L = 0.625$.

(*k, n, m*)	Sch.	C_L							
		MLE		BE:SE		BE:LINEX			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
(3, 40, 20)	S_1	0.6061	0.0087	0.7755	0.0227	0.7773	0.0233	0.7761	0.0229
	S_2	−0.6033	0.0089	0.8169	0.037	0.8182	0.0374	0.8173	0.0371
	S_3	0.6027	0.0096	0.7847	0.0256	0.7863	0.0261	0.7852	0.0257
	S_4	0.6074	0.0086	0.828	0.0414	0.8291	0.0418	0.8284	0.0415
(3, 40, 30)	S_1	0.61	0.0057	0.7692	0.0208	0.7705	0.0212	0.7696	0.021
	S_2	0.6133	0.0056	0.7983	0.0301	0.7993	0.0304	0.7986	0.0302

Table 9. (*Continued*).

(*k*, *n*, *m*)	**Sch.**	C_L							
		MLE		**BE:SE**		**BE:LINEX**			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
	S_3	0.6119	0.0058	0.7754	0.0227	0.7767	0.023	0.7759	0.0228
	S_4	0.6112	0.0058	0.8103	0.0344	0.8112	0.0347	0.8106	0.0345
(3, 80, 40)	S_1	0.6147	0.0043	0.7662	0.02	0.7672	0.0202	0.7665	0.0201
	S_2	0.6135	0.0039	0.8169	0.0369	0.8175	0.0371	0.8171	0.037
	S_3	0.6119	0.0042	0.7777	0.0233	0.7786	0.0236	0.778	0.0234
	S_4	0.615	0.0041	0.8284	0.0414	0.8289	0.0417	0.8286	0.0415
(3, 80, 60)	S_1	0.6196	0.0025	0.7619	0.0188	0.7626	0.0189	0.7621	0.0188
	S_2	0.6189	0.0027	0.7964	0.0294	0.7969	0.0296	0.7966	0.0295
	S_3	0.6201	0.0023	0.769	0.0208	0.7697	0.0209	0.7692	0.0208
	S_4	0.6172	0.0024	0.8093	0.034	0.8098	0.0342	0.8095	0.0341
(5, 40, 20)	S_1	0.5998	0.0104	0.775	0.0226	0.7768	0.0231	0.7756	0.0227
	S_2	0.606	0.0097	0.8175	0.0372	0.8187	0.0377	0.8179	0.0374
	S_3	0.6044	0.0092	0.7797	0.024	0.7814	0.0245	0.7802	0.0242
	S_4	0.6054	0.0089	0.8278	0.0413	0.8289	0.0417	0.8281	0.0414
(5, 40, 30)	S_1	0.615	0.0048	0.7694	0.0209	0.7707	0.0213	0.7698	0.021
	S_2	0.6089	0.0053	0.7977	0.0299	0.7987	0.0302	0.798	0.03
	S_3	0.6084	0.0058	0.7723	0.0217	0.7736	0.0221	0.7727	0.0219
	S_4	0.6104	0.0057	0.8098	0.0342	0.8107	0.0346	0.8101	0.0344
(5, 80, 40)	S_1	0.6191	0.0035	0.7663	0.02	0.7673	0.0203	0.7666	0.0201
	S_2	0.6147	0.0043	0.8171	0.037	0.8177	0.0372	0.8173	0.037
	S_3	0.617	0.0042	0.7727	0.0218	0.7736	0.0221	0.773	0.0219
	S_4	0.6174	0.004	0.8286	0.0415	0.8291	0.0418	0.8288	0.0416
(5, 80, 60)	S_1	0.6153	0.0026	0.7617	0.0187	0.7624	0.0189	0.7619	0.0188
	S_2	0.6186	0.0024	0.7963	0.0294	0.7968	0.0295	0.7964	0.0294
	S_3	0.6215	0.0027	0.7661	0.0199	0.7667	0.0201	0.7663	0.02
	S_4	0.6169	0.0026	0.8094	0.034	0.8098	0.0342	0.8095	0.0341

Table 10. Avg. estimated values and MSEs of the ML and Bayesian for different PFFC schemes for Pareto distribution with $\theta = 1.5$, $L = 1$ and true value of $C_L = -0.5$.

(*k*, *n*, *m*)	**Sch.**	C_L							
		MLE		**BE:SE**		**BE:LINEX**			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
(3, 40, 20)	S_1	−0.6007	0.1664	0.0863	0.3449	0.116	0.3805	0.0965	0.3569
	S_2	−0.5759	0.1554	0.2606	0.5809	0.2802	0.6109	0.2673	0.5911
	S_3	−0.5825	0.1471	0.1249	0.3917	0.1522	0.4264	0.1342	0.4034
	S_4	−0.5785	0.1416	0.3022	0.646	0.3197	0.6742	0.3081	0.6556
(3, 40, 30)	S_1	−0.54	0.0759	0.0676	0.3227	0.0885	0.3468	0.0747	0.3308
	S_2	−0.5646	0.0846	0.1829	0.4674	0.199	0.4896	0.1883	0.4748

Table 10. (*Continued*).

(*k*, *n*, *m*)	Sch.	C_L							
		MLE		BE:SE		BE:LINEX			
						$\tau = -0.5$		$\tau = 0.5$	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
	S_3	−0.5665	0.0928	0.0906	0.3494	0.1106	0.3733	0.0974	0.3575
	S_4	−0.5586	0.091	0.2321	0.5373	0.2464	0.5583	0.2369	0.5444
(3, 80, 40)	S_1	−0.5237	0.0567	0.0578	0.3115	0.074	0.3298	0.0633	0.3176
	S_2	−0.5413	0.0683	0.2636	0.5841	0.2735	0.5994	0.2669	0.5893
	S_3	−0.532	0.067	0.1043	0.3657	0.119	0.3836	0.1093	0.3717
	S_4	−0.5306	0.0639	0.31	0.6573	0.3187	0.6715	0.3129	0.6621
(3, 80, 60)	S_1	−0.5388	0.0423	0.0414	0.2932	0.0526	0.3055	0.0451	0.2973
	S_2	−0.5257	0.0378	0.181	0.4643	0.1893	0.4755	0.1838	0.468
	S_3	−0.5141	0.0424	0.0712	0.3264	0.0818	0.3386	0.0747	0.3305
	S_4	−0.5325	0.0413	0.2339	0.5392	0.2411	0.5498	0.2363	0.5428
(5,40, 20)	S_1	-0.5912	0.158	0.0842	0.3424	0.1141	0.378	0.0945	0.3544
	S_2	−0.5806	0.1502	0.2597	0.5795	0.2794	0.6095	0.2664	0.5896
	S_3	−0.5749	0.142	0.1061	0.3685	0.1345	0.4037	0.1158	0.3804
	S_4	−0.5696	0.136	0.3029	0.6472	0.3204	0.6753	0.3089	0.6567
(5, 40, 30)	S_1	−0.5585	0.0895	0.0658	0.3206	0.0868	0.3448	0.0729	0.3288
	S_2	−0.5437	0.0878	0.185	0.4703	0.2011	0.4925	0.1905	0.4778
	S_3	−0.5593	0.0943	0.0796	0.3365	0.1	0.3605	0.0865	0.3446
	S_4	−0.5449	0.0786	0.2353	0.542	0.2495	0.5629	0.2401	0.549
(5, 80, 40)	S_1	−0.5351	0.0607	0.0564	0.31	0.0727	0.3283	0.0619	0.3161
	S_2	−0.5336	0.0645	0.2638	0.5845	0.2738	0.5997	0.2672	0.5896
	S_3	−0.555	0.0673	0.0823	0.3394	0.0977	0.3575	0.0875	0.3455
	S_4	−0.5332	0.0647	0.3098	0.6571	0.3185	0.6713	0.3127	0.6618
(5, 80, 60)	S_1	−0.5336	0.0429	0.0415	0.2933	0.0527	0.3057	0.0453	0.2975
	S_2	−0.5374	0.0397	0.1805	0.4636	0.1888	0.4749	0.1833	0.4674
	S_3	−0.5274	0.0409	0.0574	0.3109	0.0684	0.3232	0.0611	0.315
	S_4	−0.5125	0.0399	0.2365	0.5431	0.2437	0.5537	0.2389	0.5467

Under the PFFC scheme, the average values and MSEs of $\widehat{C_L}$, $\hat{C}_{L_{LX}}$ and $\hat{C}_{L_{SE}}$ are listed in **Tables 7–10**. These tables lead us to the following conclusions:

- The bias of MLE is smaller than $\hat{C}_{L_{LX}}$ and $\hat{C}_{L_{SE}}$, that is the bias of MLE of C_L, is smaller than the corresponding BE under SE and LINEX loss functions.
- The bias in MLE decreases as n and m increase for the two cases when $k = 3$ or $k = 5$ except few cases.
- The MSE of the MLE ($\widehat{C_L}$) decreases as n and m increase, indicating that the MLE of C_L is consistent.
- The MSE of BE under SE and LINEX loss functions take consistent behavior when C_L is small this also when θ and L are large.
- The MSE of BE under SE and LINEX loss functions take inconsistent behavior when C_L is large this also when θ and L are small.
- In approximately most situations, the MLE of C_L is preferred over the BE under SE and LINEX

functions.

7.2. Real data analysis

A real data set is analyzed for illustrative purpose as well as to assess the statistical performances of the MLEs and Bayesian estimation for lifetime index given for Pareto distribution under different progressive type-II censoring schemes.

The following data represents wage data (in multiples of 100 US dollars) of a random sample of 30 production-line workers in a large industrial firm. The data are reported as follows:

101	103	103	104	104	105	106	107	108	111	112	112	112	115	115
116	119	119	119	123	125	128	132	140	151	154	156	157	158	198

This dataset was analyzed by Renjini et al.[25] and found that it's fitted to Pareto distribution where the Kolmogorov-Smirnov statistic and P-value are obtained as 0.0947175 and 0.927333 respectively. **Table 11** presents various estimates of C_L for different PTIIC schemes at the specified L values.

Table 11. Estimated values of the ML and Bayesian for different schemes of progressive type-II censoring at life index L for Pareto distribution given real dataset.

Sch.	MLE	BE:SE		BE:LINEX			
				$\tau = -0.5$		$\tau = 0.5$	
		IP	Non-IP	IP	Non-IP	IP	Non-IP
$L = 0.25$							
Sch.1	0.96648	0.93510	0.93475	0.93526	0.93516	0.93491	0.93480
Sch.2	0.96505	0.92949	0.92922	0.92967	0.92955	0.92941	0.92928
Sch.3	0.96490	0.92889	0.92864	0.92908	0.92895	0.92882	0.92870
Sch.4	0.94795	0.78891	0.79652	0.79057	0.78947	0.79803	0.79703
$L = 0.5$							
Sch.1	0.93296	0.87000	0.86929	0.87063	0.87021	0.86992	0.86950
Sch.2	0.93010	0.85873	0.85820	0.85947	0.85898	0.85893	0.85844
Sch.3	0.92981	0.85753	0.85702	0.85829	0.85778	0.85777	0.85727
Sch.4	0.89590	0.57558	0.59100	0.58224	0.57783	0.59704	0.59304
$L = 0.75$							
Sch.1	0.89944	0.80468	0.80363	0.80610	0.80515	0.80503	0.80410
Sch.2	0.89515	0.78771	0.78693	0.78939	0.78828	0.78858	0.78748
Sch.3	0.89471	0.78591	0.78516	0.78762	0.78648	0.78683	0.78572
Sch.4	0.84385	0.35994	0.38340	0.37498	0.36506	0.39704	0.38804

8. Concluding remarks

The LPI is effective in testing the performance of any process, in this paper we depend on the ML as a non-Bayesian technique as well as the Bayesian technique using SE and LINEX loss functions. Under Pareto distribution, using GOS samples, C_L was estimated. The power function of the test under ML and Bayesian methods was calculated and we used it to find $(1-\alpha)$ one-sided CI for C_L to determine whether LPI meets the required level or not. Additionally, to make sure our technique works as predicted, a simulated study and real data analysis were completed under the PTIIC and PFFC schemes. The simulation study showed that the MLE of C_L is superior to the Bayes estimate under SE and LINEX loss functions in both PTIIC and PFFC schemes. Moreover, the MLE based on scheme S_3 is preferable to others in the case of PTIIC. Generally speaking, the MLE is superior to other estimates according to it is consistent.

Author contributions

Conceptualization, ASH and EAE; methodology, AMF; software, AMF; validation, ASH, EAE and AMF; formal analysis, AMF; investigation, ASH; resources, EAE; data curation, AMF; writing—original draft preparation, ASH and AMF; writing—review and editing, ASH, EAE and AMF; visualization, ASH and EAE; supervision, ASH and EAE. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no conflict of interest.

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